Oral Exam for Teams: Applied and Computational Mathematics 2017

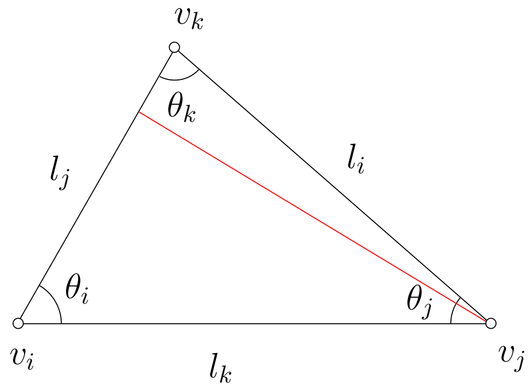

Figure 1: reference triangle.

1. Given a Euclidean triangle $[v_i, v_j, v_k]$ with edge lengths l_i, l_j, l_k and corner angles $\theta_i, \theta_j, \theta_k$ (see Figure 1), we treat the angles as the functions of edge lengths, namely, $\theta_i = \theta_i(l_i, l_j, l_k)$.

   (a) Show that
   $$\frac{\partial \theta_i}{\partial l_i} = \frac{l_i}{2A}, \quad \frac{\partial \theta_i}{\partial l_j} = -\frac{l_i}{2A}\cos\theta_k,$$
   where A is the area of the triangle.

   (b) Suppose the initial edge lengths are (l_i^0, l_j^0, l_k^0), the conformal factor (u_i, u_j, u_k) are three real numbers associated with the vertices, the vertex scaling operator changes each edge length by multiplying the exponential of conformal factors at its two end vertces, namely:
   $$l_i = e^{u_j} l_i^0 e^{u_k}, \quad l_j = e^{u_k} l_j^0 e^{u_i}, \quad l_k = e^{u_i} l_k^0 e^{u_j},$$
   Show that
   $$\frac{\partial \theta_i}{\partial u_j} = \frac{\partial \theta_j}{\partial u_i} = \cot\theta_k, \quad \frac{\partial \theta_i}{\partial u_i} = -\cot\theta_j - \cot\theta_k$$

   (c) If the initial triangle is an acute triangle, then in a neighborhood of $(u_i, u_j, u_k) = (0, 0, 0)$, the mapping $\varphi : \{(u_i, u_j, u_k) | u_i + u_j + u_k = 0\} \to \{(\theta_i, \theta_j, \theta_k) | \theta_i + \theta_j + \theta_k = \pi\}$ is diffeomorphic.

2. Let $A \in \mathbb{R}^{n\times n}$ be symmetric and let $\|q_1\|_2 = 1$. Consider the following Lanczos iteration:

$$
\begin{aligned}
& r_0 = q_1, \quad \beta_0 = 1, \quad q_0 = 0, \quad k := 0 \\
& \textbf{while} \;\; \beta_k \neq 0 \\
& \quad q_{k+1} := r_k/\beta_k \\
& \quad k := k+1 \\
& \quad \alpha_k := q_k^T A q_k \\
& \quad r_k := (A - \alpha_k I)q_k - \beta_{k-1} q_{k-1} \\
& \quad \beta_k := \|r_k\|_2 \\
& \textbf{end}
\end{aligned}
$$

Let $K_n = \text{span}\{q_1, Aq_1, \cdots, A^{n-1}q_1\}$.

(a) Show that

$$AQ_k = Q_k T_k + r_k e_k^T$$

where e_k is the k-th unit vector, $Q_k = [q_1 \cdots q_k]$ and

$$T_k = \begin{bmatrix} \alpha_1 & \beta_1 & & \cdots & 0 \\ \beta_1 & \alpha_2 & \ddots & & \vdots \\ & \ddots & \ddots & \ddots & \\ \vdots & & \ddots & \ddots & \beta_{k-1} \\ 0 & \cdots & & \beta_{k-1} & \alpha_k \end{bmatrix}$$

(b) Assume that the iteration does not terminate. Show that Q_k has orthonormal columns, and that they span K_k.

(c) Show that the Lanczos iteration will stop when $k = m$, where $m = \text{rank}(K_n)$.

(d) What is the purpose of this algorithm? Briefly justify your answer.